

Two-fluid hydrodynamics for a trapped weakly-interacting Bose gas

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We derive the coupled equations of motion for the condensate (superfluid) and non-condensate (normal fluid) degrees of freedom in a trapped Bose gas at finite temperatures. Our results are based on the Hartree-Fock-Popov approximation for the time-dependent condensate wavefunction and an assumption of local equilibrium for the non-condensate atoms. In the case of a uniform weakly-interacting gas, our formalism gives a microscopic derivation of the well-known two-fluid equations of Landau. The collective modes in a parabolically trapped Bose gas include the analogue of the out-of-phase second sound mode in uniform systems.

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The low frequency dynamics of superfluid ^4He is commonly described using the two-fluid phenomenology first developed by Tisza [1] and Landau [2]. This description, later shown to be a consequence of a Bose broken symmetry [3,4], accounts for the characteristic features associated with superfluidity in terms of the relative motion of normal fluid and superfluid degrees of freedom. In particular, it predicts the existence of second sound as an out-of-phase oscillation of the two components. In the present letter, we give a microscopic derivation of the analogous two-fluid equations for an *inhomogeneous* weakly-interacting gas of trapped atoms. In this situation, the superfluid is identified with the condensate atoms as described by a macroscopic wavefunction, while the normal fluid density is associated with the non-condensate thermal cloud. In the uniform density limit, we show that our equations are consistent with the standard two-fluid equations [2,3]. However, the hydrodynamic behavior of Bose gases is quite different from that of a Bose liquid such as superfluid ^4He .

Our analysis is based on the equation of motion of the macroscopic Bose wavefunction [3,4], $\Phi(\mathbf{r}, t)$, as determined within the time-dependent Hartree-Fock-Popov (HFP) approximation, which is a generalization of Refs. [5] and [6]. This condensate wavefunction is coupled to the thermally excited atoms making up the non-condensate which we assume is described by a semiclassical phase space distribution function $f(\mathbf{r}, \mathbf{p}, t)$. The further assumption that collisions are sufficiently rapid to force local equilibrium within the thermal cloud leads to a set of hydrodynamic conservation laws for the non-condensate component. Our final result is a set of equations for the condensate and non-condensate degrees of freedom which extends recent work on the hydrodynamics of trapped Bose gases [7] to below T_{BEC} .

We first consider the dynamics of the condensate. As usual, the Bose field operator is conveniently separated into condensate and non-condensate parts: $\hat{\psi}(\mathbf{r}) = \Phi(\mathbf{r}) + \tilde{\psi}(\mathbf{r})$. For an arbitrary nonequilibrium state, the spatially and time-varying macroscopic wavefunction $\Phi(\mathbf{r}, t) \equiv \langle \hat{\psi}(\mathbf{r}) \rangle_t$ is described within the time-dependent Hartree-Fock-Popov approximation by the equation of motion

$$i\hbar \frac{\partial \Phi(\mathbf{r}, t)}{\partial t} = \left[-\frac{\hbar^2 \nabla^2}{2m} + U_{ext}(\mathbf{r}) + 2g\tilde{n}(\mathbf{r}, t) + g n_c(\mathbf{r}, t) \right] \times \Phi(\mathbf{r}, t) \equiv \hat{\mathcal{H}}(\mathbf{r}, t) \Phi(\mathbf{r}, t). \quad (1)$$

Here the nonequilibrium non-condensate density is given by $\tilde{n}(\mathbf{r}, t) = \langle \tilde{\psi}^\dagger(\mathbf{r}) \tilde{\psi}(\mathbf{r}) \rangle_t$, the condensate density is $n_c(\mathbf{r}, t) = |\Phi(\mathbf{r}, t)|^2$ and $g = 4\pi a \hbar^2 / m$ is the interaction strength. Eq.(1) represents a natural extension of recent work [5,6] which approximated $\tilde{n}(\mathbf{r}, t)$ in (1) by the equilibrium value $\tilde{n}_0(\mathbf{r})$, thereby ignoring the collective behavior of the non-condensate. In contrast, the time-dependent condensate wavefunction $\Phi(\mathbf{r}, t)$ in (1) is coupled into the fluctuations of the non-condensate and a dynamical equation for the latter is also required.

It is convenient to recast the condensate equation of motion into a pair of hydrodynamic equations using the amplitude and phase representation $\Phi(\mathbf{r}, t) = |\Phi(\mathbf{r}, t)| e^{i\theta(\mathbf{r}, t)}$. Substituting this form into (1) and separating real and imaginary parts, one finds

$$\begin{aligned} \frac{\partial n_c}{\partial t} &= -\nabla \cdot (n_c \mathbf{v}_c) \\ m \left[\frac{\partial \mathbf{v}_c}{\partial t} + \frac{1}{2} \nabla \mathbf{v}_c^2 \right] &= -\nabla \phi, \end{aligned} \quad (2)$$

where $\mathbf{v}_c(\mathbf{r}, t) \equiv \hbar \nabla \theta(\mathbf{r}, t) / m$. The potential $\phi(\mathbf{r}, t)$ is defined by

$$\phi(\mathbf{r}, t) \equiv \frac{1}{|\Phi(\mathbf{r}, t)|} \hat{\mathcal{H}}(\mathbf{r}, t) |\Phi(\mathbf{r}, t)|, \quad (3)$$

where $\hat{\mathcal{H}}(\mathbf{r}, t)$ is the HFP Hamiltonian given in (1). Anticipating the identification of \mathbf{v}_c with the superfluid velocity \mathbf{v}_s , we see that ϕ plays the role of the chemical potential associated with the superfluid motion [3].

We next consider the dynamics of the non-condensate in the low-frequency collision-dominated hydrodynamic regime. In the semi-classical limit valid at finite temperatures [8] (with $k_B T \gg \hbar \omega_0$, $g n_0(\mathbf{r})$, where ω_0 is a characteristic trap frequency), the dynamics can be formulated in terms of a quantum kinetic equation for the distribution function $f(\mathbf{r}, \mathbf{p}, t)$ [9,10]. Since (2) implies that the number of particles in the condensate is conserved, we must for consistency exclude those processes which scatter atoms in and out of the condensate [10]. In this situation, only collisions between excited atoms are relevant and we can use the kinetic equation [9]

$$\left[\frac{\partial}{\partial t} + \frac{\mathbf{p}}{m} \cdot \nabla_{\mathbf{r}} - \nabla U(\mathbf{r}, t) \cdot \nabla_{\mathbf{p}} \right] f(\mathbf{r}, \mathbf{p}, t) = \left. \frac{\partial f}{\partial t} \right|_{coll} \quad (4)$$

Here $U(\mathbf{r}, t) \equiv U_{ext}(\mathbf{r}) + 2g[\tilde{n}(\mathbf{r}, t) + n_c(\mathbf{r}, t)]$ includes the self-consistent Hartree-Fock dynamic mean field in which the condensate part $2g n_c(\mathbf{r}, t)$ can be viewed as an additional external field acting on the non-condensate.

The required hydrodynamic equations are obtained from (4) by making the further assumption that collisions enforce the distribution function to take the local equilibrium form [9]

$$f_0(\mathbf{r}, \mathbf{p}, t) = \frac{1}{\exp \left[\beta \left\{ \frac{1}{2m} [\mathbf{p} - m\mathbf{v}_n]^2 + U - \mu \right\} \right] - 1}, \quad (5)$$

where the thermodynamic variables β , \mathbf{v}_n and μ , together with U , all depend on \mathbf{r} and t . When (5) is substituted into (4), the collision integral vanishes by virtue of the local equilibrium form of the distribution function. Taking moments of the resulting equation with respect to 1 , p_μ and p^2 , we obtain the closed set of equations [11]

$$\begin{aligned} \frac{\partial \tilde{n}}{\partial t} + \nabla \cdot (\tilde{n} \mathbf{v}_n) &= 0 \\ m \tilde{n} \left[\frac{\partial \mathbf{v}_n}{\partial t} + (\mathbf{v}_n \cdot \nabla) \mathbf{v}_n \right] &= -\nabla \tilde{P} - \tilde{n} \nabla U \\ \frac{\partial \tilde{\epsilon}}{\partial t} + \frac{5}{3} \nabla \cdot (\tilde{\epsilon} \mathbf{v}_n) &= \mathbf{v}_n \cdot \nabla \tilde{P}. \end{aligned} \quad (6)$$

The non-condensate density $\tilde{n}(\mathbf{r}, t)$ is given by

$$\tilde{n}(\mathbf{r}, t) \equiv \int \frac{d^3 p}{h^3} f_0(\mathbf{r}, \mathbf{p}, t) = \frac{1}{\Lambda^3} g_{3/2}(z(\mathbf{r}, t)), \quad (7)$$

with $z(\mathbf{r}, t) \equiv e^{\beta(\mathbf{r}, t)[\mu(\mathbf{r}, t) - U(\mathbf{r}, t)]}$ and $\Lambda(\mathbf{r}, t) = (2\pi\hbar^2/mk_B T(\mathbf{r}, t))^{1/2}$. The quantity $\tilde{\epsilon}$ is the non-convective part of the kinetic energy density defined with $\mathbf{v}_n = 0$

in (5). Similarly, $\tilde{P} = \frac{2}{3}\tilde{\epsilon}$ is the kinetic contribution to the local equilibrium pressure defined by

$$\begin{aligned} \tilde{P}(\mathbf{r}, t) &\equiv \int \frac{d^3 p}{h^3} \frac{p^2}{3m} f_0(\mathbf{r}, \mathbf{p}, t) \Big|_{\mathbf{v}_n=0} \\ &= \frac{1}{\beta \Lambda^3} g_{5/2}(z(\mathbf{r}, t)). \end{aligned} \quad (8)$$

Eqs. (2) and (6) constitute our full set of nonlinear hydrodynamic equations for a trapped Bose gas at finite temperatures.

At the level of approximation we are considering, the condensate and non-condensate satisfy separate continuity equations. Combining these two equations gives the expected two-fluid continuity equation

$$\frac{\partial n}{\partial t} = -\nabla \cdot \mathbf{j}, \quad (9)$$

where $n \equiv \tilde{n} + n_c$ and $\mathbf{j} \equiv \tilde{n} \mathbf{v}_n + n_c \mathbf{v}_c$. We identify the normal fluid density with $\tilde{n}(\mathbf{r}, t)$ and the superfluid density with $n_c(\mathbf{r}, t)$. The former identification is supported by noting that (7) can be expressed equivalently as

$$\tilde{n}(\mathbf{r}, t) = - \int \frac{d^3 p}{h^3} \frac{p^2}{3m} \frac{\partial f_0(\varepsilon_p)}{\partial \varepsilon_p}, \quad (10)$$

where $\varepsilon_p(\mathbf{r}, t) \equiv \frac{p^2}{2m} + U(\mathbf{r}, t) - \mu(\mathbf{r}, t)$ is the excitation energy. This is the usual Landau formula for the normal fluid density [2,3].

The linearized version of (2) and (6) allows one to consider small amplitude oscillations about equilibrium. The equilibrium condensate wavefunction is determined by the solution of

$$\begin{aligned} \hat{\mathcal{H}}_0(\mathbf{r}) \Phi_0(\mathbf{r}) &\equiv \left[-\frac{\hbar^2 \nabla^2}{2m} + U_{ext}(\mathbf{r}) + 2g\tilde{n}_0(\mathbf{r}) \right. \\ &\quad \left. + g n_{c0}(\mathbf{r}) \right] \Phi_0(\mathbf{r}) = \mu_0 \Phi_0(\mathbf{r}), \end{aligned} \quad (11)$$

with $n_{c0}(\mathbf{r}) = |\Phi_0(\mathbf{r})|^2$. The equilibrium non-condensate density $\tilde{n}_0(\mathbf{r})$ is given by (7) with the equilibrium fugacity defined as $z_0 = e^{\beta_0[\mu_0 - U_0(\mathbf{r})]}$, with $U_0(\mathbf{r}) = U_{ext}(\mathbf{r}) + 2g n_0(\mathbf{r})$. Eq.(11) and the equilibrium version of (7) must be solved self-consistently.

The linearization of (2) around equilibrium leads to the condensate equations

$$\begin{aligned} \frac{\partial \delta n_c}{\partial t} &= -\nabla \cdot (n_{c0} \delta \mathbf{v}_c) \\ m \frac{\partial \delta \mathbf{v}_c}{\partial t} &= -\nabla \delta \phi, \end{aligned} \quad (12)$$

where

$$\begin{aligned} \delta \phi(\mathbf{r}, t) &\equiv \frac{1}{|\Phi_0(\mathbf{r})|} [\hat{\mathcal{H}}_0(\mathbf{r}) - \mu_0] \delta |\Phi(\mathbf{r}, t)| \\ &\quad + g \delta n_c(\mathbf{r}, t) + 2g \delta \tilde{n}(\mathbf{r}, t). \end{aligned} \quad (13)$$

In arriving at this result, we have noted that $\nabla\phi_0(\mathbf{r}) = 0$ in equilibrium. Similarly, the linearization of (6) leads to the equations

$$\begin{aligned}\frac{\partial\delta\tilde{n}}{\partial t} &= -\nabla \cdot (\tilde{n}_0\delta\mathbf{v}_n) \\ m\tilde{n}_0\frac{\partial\delta\mathbf{v}_n}{\partial t} &= -\nabla\delta\tilde{P} - \delta\tilde{n}\nabla U_0 - 2g\tilde{n}_0\nabla(\delta\tilde{n} + \delta n_c) \\ \frac{\partial\delta\tilde{P}}{\partial t} &= -\frac{5}{3}\nabla \cdot (\tilde{P}_0\delta\mathbf{v}_n) + \frac{2}{3}\delta\mathbf{v}_n \cdot \nabla\tilde{P}_0,\end{aligned}\quad (14)$$

where $\tilde{P}_0(\mathbf{r})$ is the equilibrium kinetic pressure which satisfies $\nabla\tilde{P}_0 = -\tilde{n}_0\nabla U_0$. Above T_{BEC} , these equations reduce to those of Ref. [7] if we ignore the effect of interactions ($g = 0$).

For its intrinsic interest, and in order to better understand the implications of our two-fluid equations, we now consider the limit of a *homogeneous* system ($U_{ext}(\mathbf{r}) = 0$). In this special case, (11) yields a uniform condensate with the chemical potential having the Thomas-Fermi (TF) value $\mu_0 = 2g\tilde{n}_0 + gn_{c0}$. Taking all equilibrium quantities to be spatially-independent, and noting that the first term on the right hand side of (13) can be neglected in the long-wavelength limit, the two velocity equations reduce to

$$\begin{aligned}m\frac{\partial\delta\mathbf{v}_c}{\partial t} &= -2g\nabla\delta\tilde{n} - g\nabla\delta n_c \\ m\tilde{n}_0\frac{\partial\delta\mathbf{v}_n}{\partial t} &= -\nabla\delta\tilde{P} - 2g\tilde{n}_0\nabla\delta\tilde{n} - 2g\tilde{n}_0\nabla\delta n_c,\end{aligned}\quad (15)$$

with

$$\frac{\partial\delta\tilde{P}}{\partial t} = -\frac{5}{3}\tilde{P}_0\nabla \cdot \delta\mathbf{v}_n. \quad (16)$$

The two equations in (15) can be combined to give

$$\begin{aligned}\frac{\partial}{\partial t}\delta\mathbf{j} &= -\nabla\delta\tilde{P} - 2g(n_{c0} + \tilde{n}_0)\nabla\delta\tilde{n} - g(n_{c0} + 2\tilde{n}_0)\nabla\delta n_c \\ &\equiv -\nabla\delta P.\end{aligned}\quad (17)$$

One can verify that (17) is consistent with the following expression for the *total* local thermodynamic pressure

$$P = \tilde{P} + \frac{1}{2}g[n^2 + 2n\tilde{n} - \tilde{n}^2], \quad (18)$$

which is the equation of state at the level of approximation we are considering. Within the same approximation, the total internal energy density is given by

$$\begin{aligned}\epsilon(\mathbf{r}, t) &= \tilde{\epsilon}(\mathbf{r}, t) + \frac{1}{2}g\langle\hat{\psi}^\dagger(\mathbf{r})\hat{\psi}^\dagger(\mathbf{r})\hat{\psi}(\mathbf{r})\hat{\psi}(\mathbf{r})\rangle_t \\ &\simeq \tilde{\epsilon}(\mathbf{r}, t) + \frac{1}{2}g[n^2 + 2n\tilde{n} - \tilde{n}^2].\end{aligned}\quad (19)$$

These results and the equilibrium thermodynamic relation

$$\epsilon + P = sT + \mu n, \quad (20)$$

allow us to identify the entropy density s . Using $\mu_0 = g(2\tilde{n}_0 + n_{c0})$ together with (18) and (19), (20) gives

$$s_0T_0 = \frac{5}{2}\tilde{P}_0 + gn_{c0}\tilde{n}_0. \quad (21)$$

Here \tilde{P}_0 is the equilibrium kinetic pressure defined in (8), with $z \equiv z_0 = e^{\beta_0(\mu_0 - 2gn_0)} = e^{-\beta gn_{c0}}$. Using the local equilibrium expression for \tilde{P} in (8), one finds that the fluctuation in the total pressure P is given by

$$\delta P = s_0\delta T + \tilde{n}_0\delta\mu + gn_{c0}(2\delta\tilde{n} + \delta n_c), \quad (22)$$

with s_0 defined by (21). Comparing this to the thermodynamic relation $\delta P = s_0\delta T + n_0\delta\mu$, we arrive at

$$\delta\mu = \delta(2g\tilde{n} + gn_c). \quad (23)$$

This result confirms (in the case of a uniform equilibrium density) that $\delta\phi(\mathbf{r}, t)$ in (12) and (13) is indeed the fluctuation in the local chemical potential. More generally, we have verified that (2) is equivalent to the key Landau equation for superfluid flow [2–4]

$$m\left[\frac{\partial\mathbf{v}_S}{\partial t} + \frac{1}{2}\nabla\mathbf{v}_S^2\right] = -\nabla\mu(\mathbf{r}, t). \quad (24)$$

Finally, using (19) we find

$$\delta\epsilon = \frac{3}{2}\delta\tilde{P} + gn_{c0}(2\delta\tilde{n} + \delta n_c) + 2g\tilde{n}_0\delta n. \quad (25)$$

Inserting this result into the thermodynamic relation $\delta\epsilon = T_0\delta s + \mu_0\delta n$, a simple calculation gives

$$T_0\delta s = \frac{3}{2}\delta\tilde{P} + gn_{c0}\delta\tilde{n}. \quad (26)$$

Taking the time-derivative, and using (16) and the continuity equation for $\delta\tilde{n}(\mathbf{r}, t)$, we obtain the linearized form of Landau's entropy conservation equation [2–4]

$$\frac{\partial\delta s}{\partial t} = -s_0\nabla \cdot \delta\mathbf{v}_n. \quad (27)$$

In the uniform case, Eqs. (15) and (16) are easily solved to give the expected first and second sound phonon modes. In contrast to superfluid ^4He , second sound in a gas involves a condensate oscillation largely uncoupled from the non-condensate, with a velocity given by $u_2 = (gn_{c0}/m)^{1/2}$.

As a specific application to a trapped Bose gas, we consider the center-of-mass mode solution for an anisotropic parabolic potential $U_{ext}(\mathbf{r}) = \frac{1}{2}m(\omega_x^2x^2 + \omega_y^2y^2 + \omega_z^2z^2)$. On the basis of the generalized Kohn theorem [12], one expects three modes in which the gas oscillates rigidly along each of the principal directions at the appropriate trap frequency ω_i . Denoting the displacement of the gas by $\boldsymbol{\eta}(t)$, the condensate and non-condensate densities *both* behave as $n_0(\mathbf{r} - \boldsymbol{\eta}(t))$, giving a density fluctuation $\delta n(\mathbf{r}, t) = -\nabla n_0(\mathbf{r}) \cdot \boldsymbol{\eta}(t)$, and a velocity field

$\mathbf{v}(\mathbf{r}, t) = \dot{\boldsymbol{\eta}}(t)$ which is spatially independent. One can easily verify that our linearized equations (12) and (14) admit a solution of this kind with both the condensate and non-condensate having identical displacements $\boldsymbol{\eta}(t) = \boldsymbol{\eta}_0 \cos \omega_i t$. Thus, in contrast to the static HFP used in Ref. [5], the dynamic HFP theory given by (1) is consistent with the generalized Kohn theorem.

An estimate of the frequency of the out-of-phase dipole mode in a trapped gas can be based on the intuitive idea that the condensate and non-condensate oscillate rigidly against each other. The problem is then equivalent to a two-particle problem with masses $M_c = mN_c$ and $M_n = m\tilde{N}$, each confined in the parabolic potential and coupled together by a spring with force constant k . Since the interaction energy between the two components is $E_{int} = 2g \int d\mathbf{r} n_c(\mathbf{r})\tilde{n}(\mathbf{r})$, k can be determined by considering small displacements of the two components along the i -th direction. The equations of motion for the two coupled masses can then be solved, giving the in-phase center-of-mass mode at the frequency ω_i discussed above, and an out-of-phase mode at the frequency

$$\Omega_i^2 = \omega_i^2 - 2g \frac{M_c + M_n}{M_c M_n} \int d\mathbf{r} \frac{\partial n_{c0}(\mathbf{r})}{\partial x_i} \frac{\partial \tilde{n}_0(\mathbf{r})}{\partial x_i}. \quad (28)$$

This result is confirmed by a more detailed treatment [13] in which the hydrodynamic equations (12–14) are cast into the form of a variational principle. In Fig. 1 we show the in- and out-of-phase mode frequencies as a function of temperature for a gas trapped in an isotropic parabolic potential. We also show the relative amplitudes of the condensate and non-condensate oscillations which satisfy the condition $M_c \eta_c + M_n \eta_n = 0$, corresponding to the center of mass being stationary for any temperature. A measurement of these amplitudes would therefore determine directly the ratio N_c/\tilde{N} of the two fluid components. This mode of the trapped Bose gas is the analogue of the usual second sound mode [3] in bulk superfluid ^4He (for which $\rho_S \mathbf{v}_S + \rho_N \mathbf{v}_N = 0$).

In summary, we have given a microscopic derivation of the hydrodynamic two-fluid equations for a trapped weakly-interacting Bose gas, as summarized by Eqs. (12) – (14). Our analysis shows how such a two-fluid description arises naturally from the existence of a macroscopic condensate wavefunction which is coupled to the non-condensate atoms. We have also shown explicitly how our linearized two-fluid equations reduce to the well-known equations (2), (9), (17) and (27) of Landau [2,3] when applied to a uniform Bose-condensed gas. Finally, we have shown that our equations are consistent with the generalized Kohn theorem and obtained the out-of-phase mode analogue of second-sound in a trapped Bose gas. A more complete discussion of the two-fluid dynamics will be given elsewhere [13].

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- [1] L. Tisza, *Nature (London)*, **141**, 913 (1938); *J. Phys. Radium*, **1**, 165 (1940).
 - [2] L.D. Landau, *J. Phys. U.S.S.R.* **5**, 71 (1941).
 - [3] P. Nozières and D. Pines, *The Theory of Quantum Liquids*, Vol. 2 (Addison-Wesley, Redwood City, CA, 1990), Chs. 7 and 10.
 - [4] P.C. Hohenberg and P.C. Martin, *Ann. Phys. (N.Y.)*, **34**, 291 (1965).
 - [5] D.A.W. Hutchinson, E. Zaremba and A. Griffin, *Phys. Rev. Lett.* **78**, 1842 (1997).
 - [6] A. Griffin, *Phys. Rev. B* **53**, 9341 (1996).
 - [7] A. Griffin, W.C. Wu and S. Stringari, *Phys. Rev. Lett.* **78**, 1838 (1997).
 - [8] S. Giorgini, L. Pitaevskii and S. Stringari, *Journ. Low Temp. Phys.*, to be published; cond-mat/9704014. These authors show that the particle-like excitation energy we use is valid to quite low temperatures in a trapped gas.
 - [9] L. P. Kadanoff and G. Baym, *Quantum Statistical Mechanics* (W.A. Benjamin, N.Y., 1962), Ch. 6.
 - [10] T.R. Kirkpatrick and J.R. Dorfman, *Journ. Low Temp. Phys.* **58**, 304 (1985); *ibid.* 399 (1985). While quite different in detail, our work may be viewed as a generalization of these papers to the case of a trapped Bose gas.
 - [11] Eq.(58) in the first paper of Ref. [10] is a kinetic equation for excitations in the local rest frame of the condensate. We have verified that this equation leads to the same set of hydrodynamic equations as in (6) if excitations in and out of the condensate are ignored.
 - [12] J.F. Dobson, *Phys. Rev. Lett.* **73**, 7244 (1994).
 - [13] E. Zaremba and A. Griffin, to be published.

FIG. 1. (a) Mode frequencies for the in-phase (solid dots) and out-of-phase (open dots) dipole modes vs. temperature for 2000 Rb atoms in an isotropic parabolic trap (see Ref. [5] for values of the physical parameters used). (b) Condensate (η_c) and non-condensate (η_n) amplitudes for the out-of-phase dipole mode. (c) Fraction of atoms in the condensate as a function of temperature.

